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Anomalous Diffusion in a Random Velocity Field

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We construct diffusions in random velocity fields which present anomalous superdiffusive behavior. The mean square displacement can be made to have any power law t^α for $1 \leqslant \alpha < 2$. Higher moments and characteristic functions are also investigated.

KEY WORDS: Diffusion in random field; superdiffusive behavior.

1. INTRODUCTION

In this note, we extend a model of anomalous diffusion in a disordered lattice introduced in ref. 1 (see also refs. 2 and 3). Let us briefly recall the result obtained in refs. 1 and 2: we have described a random motion of a particle in a d-dimensional disordered lattice, starting from 0, and we found that the mean square displacement at time n was given by

$$\langle r^2(n) \rangle \sim \begin{cases} n^{3/2} & \text{if} \quad d=2 \\ n \log n & \text{if} \quad d=3 \\ n & \text{if} \quad d \geqslant 4 \end{cases}$$

Our purpose is to give a continuous version of a more general model so that we can obtain any exponent $1 \leqslant \alpha \leqslant 2$ with

$$\langle r^2(n) \rangle \sim n^\alpha$$

and also to compute higher moments of the displacement. It is in general rather difficult to obtain rigorous analytical results for motions of particle

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in a disordered medium, due to the non-Markovian character of such motions. Many authors have used a renormalization group analysis,[4–9] but this is not a completely rigorous approach. The class of models presented below can be rigorously treated. Other rigorous scaling results in the discrete case have also been obtained in ref. 10.

A concrete example of this class of model is the diffusion of a fluid in a stratified medium. Each layer of the medium has its own transport property inducing a different velocity field of the fluid parallel to the layer. Moreover, there is a pure diffusion between layers.[2,3]

Another example of this class of model is the diffusive transport of particles in a turbulent fluid. In this case one has a statistical distribution of a velocity field which drives a passive system of particles. The diffusion is enhanced and gives the 4/3 law proposed by Richardson.[11] The moments of the scalar field were computed exactly by Kraichnan[12] in the case where the velocity field is white noise in time.

2. DESCRIPTION OF THE MODEL

We define the model in a $(d+1)$-dimensional space $\mathbb{R}^{d+1}$ with coordinates (x, y), where x is the first coordinate and $y = (y_1, \ldots, y_d)$ are the last d coordinates. On the y-space, we define a random field $\omega(y)$. Let us denote by $(x(t), y(t))$ the position of the particle at time t: $y(t)$ is a d-dimensional Brownian motion starting from 0 at $t = 0$. The particle moves with velocity $\omega(y(t))$ in the x direction, so that

$$x(t) = \int_0^t \omega(y(s))\, ds \tag{1}$$

We define

$$\begin{gathered} \langle \omega(y) \rangle = 0 \\ \langle \omega(y)\, \omega(y') \rangle = \varphi(|y - y'|) \end{gathered} \tag{2}$$

where $\langle \cdots \rangle$ is the average over the stochastic state of the field ω and $\varphi(r)$ is a given function of r. That function expresses the smoothness or self-correlation of the flow in the y direction(s). In situations where φ is a slowly decreasing function, we will see that the diffusion in x acquires anomalous properties.

We shall also denote by E the expectation over the path of the

Brownian motion $y(t)$. We want to compute the mean square displacement of the particle $(x(t), y(t))$ at time t. The main quantity is thus

$$\begin{aligned}\langle E(x(t)^2)\rangle &= \left\langle E\left(\int_0^t \omega(y(s))\,ds \int_0^t \omega(y(s'))\,ds'\right)\right\rangle \\ &= 2\int_0^t ds \int_0^s ds' \int_{\mathbb{R}^d} \varphi(y)\,\frac{e^{-|y|^2/2(s-s')}}{[2\pi(s-s')]^{d/2}}\,dy \\ &= 2\int_0^t (t-s)\,ds \int_{\mathbb{R}^d} \varphi(y)\,\frac{e^{-|y|^2/2s}}{(2\pi s)^{d/2}}\,dy \qquad (3)\end{aligned}$$

3. ASYMPTOTIC BEHAVIOR OF (3) FOR $t \to \infty$

We shall consider two different situations.

First Situation. φ is integrable. If $d=1$, when $s \to \infty$, $e^{-|y|^2/2s}$ tends to 1, and we obtain directly

$$\langle E(x(t)^2)\rangle \sim \frac{8}{3}\,\frac{1}{(2\pi)^{1/2}}\left(\int \varphi(y)\,dy\right) t^{3/2} \qquad (4)$$

On the other hand, if $d=2$, it is easy to obtain

$$\langle E(x(t)^2)\rangle \sim \frac{1}{\pi}\left(\int \varphi(y)\,dy\right) t \log t \qquad (5)$$

If $d \geqslant 3$, we observe that when t tends to ∞,

$$\int_0^t ds \int_{\mathbb{R}^d} \varphi(y)\,\frac{e^{-|y|^2/2s}}{(2\pi s)^{d/2}}\,dy \to \frac{1}{\sigma_d(d-2)} \int \varphi(y)\,\frac{dy}{|y|^{d-2}}$$

where σ_d denotes the area of the unit sphere of $\mathbb{R}^d$ and we obtain

$$\langle E(x(t)^2)\rangle \sim \frac{2}{\sigma_d(d-2)}\left(\int \varphi(y)\,\frac{dy}{|y|^{d-2}}\right) t \qquad (6)$$

Second Situation. $\varphi(|y|) \sim C/|y|^{\alpha}$ for $|y| \to \infty$. Here C is a certain constant and α is a positive exponent such that

$$\begin{aligned} &0<\alpha<1 \quad \text{if } d=1 \\ &0<\alpha<2 \quad \text{if } d\geqslant 2 \end{aligned} \qquad (7)$$

We want to study for $\beta = 0$ or 1 the asymptotic behavior of

$$\begin{aligned} I_\beta(t) &= \int_0^t s^\beta \, ds \int_{\mathbb{R}^d} \varphi(y) \frac{e^{-|y|^2/2s}}{(2\pi s)^{d/2}} \, dy \\ &= \int_0^t s^{\beta - \alpha/2} \, ds \int_{\mathbb{R}^d} \frac{e^{-|y|^2/2}}{(2\pi)^{d/2}} \frac{(s^{1/2} |y|)^\alpha \varphi(s^{1/2} |y|)}{|y|^\alpha} \, dy \\ &= \int_0^t s^{\beta - \alpha/2} \, ds \int_{\mathbb{R}^d} \frac{e^{-|y|^2/2}}{(2\pi)^{d/2}} \frac{C}{|y|^\alpha} \, dy \\ &\quad + \int_0^t s^{\beta - \alpha/2} \, ds \int_{\mathbb{R}^d} \frac{e^{-|y|^2/2}}{(2\pi)^{d/2}} \frac{1}{|y|^\alpha} K(\sqrt{s}\,|y|) \, dy \end{aligned} \tag{8}$$

where $K(r) = r^\alpha \varphi(r) - C$ and tends to 0 if r tends to infinity.

Under the hypothesis (7), the first integral in (8) is finite and has the value

$$\frac{t^{1+\beta-\alpha/2}}{1+\beta-\alpha/2} C\sigma_d \int_0^\infty \frac{e^{-r^2/2}}{(2\pi)^{d/2}} r^{d-1-\alpha} \, dr$$

and the second integral of (8) has the form

$$\int_0^t s^{\beta - \alpha/2} a(s) \, ds$$

where $a(s)$ tends to 0 if $s \to \infty$. From these remarks and from formula (3), we obtain

$$\langle E(x(t)^2) \rangle \sim \frac{2C\sigma_d}{(1-\alpha/2)(2-\alpha/2)(2\pi)^{d/2}} \left(\int_0^\infty e^{-r^2/2} r^{d-1-\alpha} \, dr \right) t^{2-\alpha/2} \tag{9}$$

In this case, we see that we can obtain any anomalous superdiffusive behavior for all possible exponents between 3/2 and 2 if $d = 1$, and between 1 and 2 if $d \geqslant 2$.

4. ANALYSIS OF THE HIGHER MOMENTS

To analyze the higher moments, we shall assume that $\omega(y)$ is a Gaussian random field with correlation φ as in (2). We have

$$\begin{aligned} \langle E(x(t)^{2N}) \rangle = (2N)! \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{2N-1}} ds_{2N} \\ \times (\langle \omega(y(s_{2N})) \, \omega(y(s_{2N-1}) \cdots \omega(y(s_1)) \rangle)) \end{aligned} \tag{10}$$

and the average over the Gaussian random field is given by the usual formula

$$\sum_{\substack{i_1 > \cdots > i_N \\ (j_1,\ldots,j_N) \\ i_1 > j_1,\ldots,i_N > j_N}} \prod_{k=1}^{N} \varphi(|y(s_{i_k}) - y(s_{j_k})|) \tag{11}$$

We shall also assume that we are in the second situation,

$$\varphi(|y|) \sim \frac{C}{|y|^\alpha} \equiv \psi_\alpha(|y|)$$

with restrictions (8) on the exponents.

If we replace all the φ in (11) by the asymptotic forms ψ_α, and define $s_j = t\sigma_j$ and $y(t\sigma_j) = \sqrt{t}\, z(\sigma_j)$ with a new Brownian motion z, we obtain

$$\left[(2N)! \int_0^1 d\sigma_{2N} \int_0^{\sigma_{2N}} d\sigma_{2N-1} \cdots \int_0^{\sigma_2} d\sigma_1 \times E\left\{\sum \prod_{k=1}^{N} \psi_\alpha(|z(\sigma_{i_k}) - z(\sigma_{j_k})|)\right\}\right] t^{(2-\alpha/2)N} \tag{12}$$

The expression in brackets will be computed exactly in the next section and in particular will be proven to be finite. Moreover, we shall estimate the characteristic function

$$F(\xi, t) = \langle E(\exp[\xi x(t)])\rangle$$

It is very easy to obtain a bound on $F(\xi, t)$ by a function with a finite radius of convergence. We want to obtain a bound by a function with an *infinite* radius of convergence. We shall now assume that the correlation function is

$$\varphi(y) = \frac{C}{|y|^\alpha}, \qquad 0 < \alpha < 1 \quad \text{if } d = 1, \qquad 0 < \alpha < 2 \quad \text{if } d \geqslant 2$$

We know from (10)–(12) that

$$\langle E(x(t)^{2N})\rangle = (2N)!\, C^{2N} t^{(2-\alpha/2)N} \int_0^1 d\sigma_{2N} \int_0^{\sigma_{2N}} d\sigma_{2N-1} \cdots \int_0^{\sigma_2} d\sigma_1 \times \sum_{I,J} E\left\{\frac{1}{|y(\sigma_{i_1}) - y(\sigma_{j_1})|^\alpha \cdots |y(\sigma_{i_N} - y(\sigma_{j_N})|^\alpha}\right\} \tag{13}$$

where I and J are varying over the sets of indices

$$I=\{i_1>\cdots>i_N\}$$
$$J=\{j_1\cdots j_N\} \text{ with } i_k>j_k \text{ for all } h$$
$$I\cup J=(1,\ldots,2N)$$

We want to compute precisely this expectation.

For a fixed partition I, J this is

$$\int \Pi\, du_i \int_0^1 d\sigma_{2N}\cdots\int_0^{\sigma_2} d\sigma_1 \frac{1}{|u_{i_1}+u_{i_1-1}+\cdots+u_{j_1+1}|^\alpha\cdots|u_{i_N}+\cdots+u_{j_N+1}|^\alpha}$$
$$\times\exp\left(-\frac{|u_{i_1}|^2}{2\,|\sigma_{i_1}-\sigma_{i_1-1}|}\right)\cdots\exp\left(-\frac{|u_{j_N+1}|^2}{2\,|\sigma_{j_N+1}-\sigma_{j_N}|}\right)$$
$$\times[2\pi(\sigma_{i_1}-\sigma_{i_1-1})\cdots(2\pi)(\sigma_{j_N+1}-\sigma_{j_N})]^{-d/2} \tag{14}$$

One of the indices j_l is 1: suppose that $j_{l_1}=1$ and reorder the indices j_l by increasing order so that now $j_1=1<\cdots<j_N$. We rescale

$$u_{j_k+1}=v_{j_k+1}(\sigma_{j_k+1}-\sigma_{j_k})^{1/2}$$

and we denote

$$K_\alpha=\sup_w\left(\int\frac{1}{|w+v|^\alpha}\frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}\,dv\right) \tag{15}$$

so that we finally obtain a bound of a term like (14) by

$$K_\alpha^n\int_0^1 d\sigma_{2N}\int_0^{\sigma_{2N}} d\sigma_{2N-1}\cdots\int_0^{\sigma_2} d\sigma_1\frac{1}{(\sigma_{j_N+1}-\sigma_{j_N})^{\alpha/2}\cdots(\sigma_2-\sigma_1)^{\alpha/2}} \tag{16}$$

Call $\gamma=\alpha/2$. An integral like (14) can be written as

$$\int_0^1\cdots\int_0^1 p_{j_3+1}^{j_3-3\gamma}\,dp_{j_3+1}\int_0^1\frac{p_{j_3}^{j_3-1-2\gamma}}{(1-p_{j_3})^\gamma}\,dp_{j_3}\cdots$$
$$\times\int_0^1 p_{j_2+1}^{j_2-2\gamma}\,dp_{j_2+1}\int_0^1\frac{p_{j_2}^{j_2-1-\gamma}}{(1-p_{j_2})^\gamma}\cdots\int_0^1 p_2^{1-\gamma}\,dp_2\int_0^1\frac{dp_1}{(1-p_1)^\gamma}$$

Let us compare this integral with the integral with the same j_k except that a particular j_l has been changed into $j_l'=j_l-1$ (we assume that this is possible or that $j_{l-1}<j_l-1$). This means that we would change

$$\int_0^1\frac{p_{j_l}^{j_l-1-(l-1)\gamma}}{(1-p_{j_l})^\gamma}\,dp_{j_l}\cdots\int_0^1 p_{j_l-1}^{j_l-2-(l-1)\gamma}\,dp_{j_l-1}$$

into

$$\int_0^1 p_{j_l}^{j_l-1-l\gamma}\,dp_{j_l}\cdots\int_0^1 \frac{p_{j_l-1}^{j_l-2-(l-1)\gamma}}{(1-p_{j_l-1})^{\gamma}}\,dp_{j_l-1}$$

The first integral is

$$\frac{1}{j_l-1-(l-1)\gamma}\,\frac{\Gamma(j_l-(l-1)\gamma)\,\Gamma(1-\gamma)}{\Gamma(j_l+1-l\gamma)}$$

and the second integral is

$$\frac{\Gamma(j_l-1-(l-1)\gamma)\,\Gamma(1-\gamma)}{(j_l-l\gamma)\,\Gamma(j_l-l\gamma)}$$

These are equal. It is then sufficient to consider an integral like (14) for $j_1=1<j_2=2<\cdots<j_N=N$, for which it is equal to

$$\int_0^1 \frac{dp_1}{(1-p_1)^{\gamma}}\int_0^1 \frac{p_2^{1-\gamma}\,dp_2}{(1-p_2)^{\gamma}}\int_0^1 \frac{p_3^{2-2\gamma}}{(1-p_3)^{\gamma}}\,dp_3\cdots\int_0^1 \frac{p_{N-1}^{(N-2)(1-\gamma)}}{(1-p_{N-1})^{\gamma}}\,dp_{N-1}$$

$$\times\int_0^1 p_N^{(N-2)(1-\gamma)+1}\,dp_N\int_0^1 p_{N+1}^{(N-2)(1-\gamma)+2}\,dp_{N+1}\cdots$$

$$\times\int_0^1 p_{2N}^{(N-2)(1-\gamma)+N+1}\,dp_{2N}$$

$$=\frac{\Gamma(1-\gamma)^{N-1}}{\Gamma(1+(N-1)(1-\gamma))}\prod_{k=2}^{N+2}\frac{1}{k+(N-2)(1-\gamma)}$$

$$=\frac{\Gamma(1-\gamma)^{N-1}}{\Gamma(1+(N-1)(1-\gamma))}\,\frac{\Gamma((N-2)(1-\gamma)+2)}{\Gamma((N-2)(1-\gamma)+N+3)}$$

The number of partitions (I, J) is

$$(2N-1)(2N-3)\cdots 3\cdot 1=2^N\,\frac{\Gamma(1/2+N)}{\Gamma(1/2)}$$

so that finally

$$\langle E(x(t)^{2N})\rangle \leqslant (2N)!\,(2C^2K_\alpha t^{2-\alpha/2})^N\,\frac{\Gamma(1-\gamma)^{N-1}}{\Gamma(1/2)}$$

$$\times\frac{\Gamma(1/2+N)\,\Gamma((N-2)(1-\gamma)+2)}{\Gamma(1+(N-1)(1-\gamma))\,\Gamma((N-2)(1-\gamma)+N+3)} \tag{17}$$

Let us now define the generalized hypergeometric function:

$$\Phi_\gamma(x) = \sum_{N\geqslant 0} x^N \frac{\Gamma(1/2+N)\,\Gamma(N(1-\gamma)+2\gamma)}{\Gamma(N(1-\gamma)+\gamma)\,\Gamma(N(2-\gamma)+2\gamma+1)}$$

Then we have the upper bound in the sense of majorant function theory:

$$\langle E(\exp[\xi x(t)])\rangle \ll \frac{1}{\Gamma(1/2)\,\Gamma(1-\gamma)}\,\Phi_\gamma(2\xi^2 C^2 K_\alpha t^{2-\gamma}\Gamma(1-\gamma)) \tag{18}$$

where $\gamma = \alpha/2$. Because $\gamma = \alpha/2 < 1$, it is clear that Φ_γ has an infinite radius of convergence, but it is not a classical hypergeometric function. The sign $\ll$ means that each Taylor coefficient of the first member is less than the corresponding Taylor coefficient of the second member.

5. ESTIMATION OF THE DISTRIBUTION OF $x(t)$

From (14), one can deduce an asymptotic bound of the rescaled distribution of $x(t)$. More precisely, the second member of (17) is bounded by

$$\langle E(x(t)^{2N})\rangle \leqslant (t^{1-\gamma/2})^{2N}\,(A_{d,\gamma,c})^{2N}\,\Gamma((1+\gamma)N-\gamma+\tfrac{1}{2}) \tag{19}$$

where $A_{d,\gamma,c}$ is a constant depending only on d, γ, and C, namely,

$$A_{d,\gamma,c} = [A(1-\gamma)]^{1/2}\,2C(2K_\alpha)^{1/2}\,(2-\gamma)^{-1+\gamma/2}\,(1+\gamma)^{-1/2-\gamma/2}$$

Let us now define the rescaled variable

$$y(t) = \left(\frac{x(t)}{t^{1-\gamma/2}A_{d,\gamma,c}}\right)^2$$

Then, using (19), for large y, the distribution probability of $y(t)$ is bounded from above by

$$\text{Prob}(y(t)\in dy) \leqslant B\exp(-y^{1/(1+\gamma)})\,y^{-(2\gamma+1/2)/(1+\gamma)}\,dy$$

Remark. Proof that K_α is bounded. We consider

$$\int \frac{1}{|w-v|^\alpha}\frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}\,dv = \int_{|v-w|<r}\frac{1}{|w-v|^\alpha}\frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}\,dv + \int_{|v-w|>r}\frac{1}{|w-v|^\alpha}\frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}\,dv$$

The first integral is less that $Cr^{d-\alpha}$ and can be made $<\varepsilon$ for r sufficiently small. Then in the second integral, the integrand tends to 0 if $|w| \to \infty$, while it is dominated by the integrable function

$$\frac{1}{r^{\alpha}} \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}$$

and by the Lebesgue theorem can be made less than ε for $|w|$ large enough.

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